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THE CLASSIFICATION OF FLAT SOLVMANIFOLDS

BY

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Dedicated to my mother and to J. Grady Horne

ABSTRACT. This paper contains a complete algebraic characterization of the fundamental groups of flat solvmanifolds. This characterization is in terms of finite integral representations of free abelian groups and the associated cohomology. A classification of compact flat solvmanifolds follows, and a list of all compact flat solvmanifolds of dimensions 3, 4, and 5 (except the 5-dimensional with first betti number 1) is given.

Some theorems on the classification of noncompact flat solvmanifolds have also been obtained. These give full results in some cases, partial results in others. For example, the odd order holonomy group case is completely settled.

I. Introduction. This paper is concerned with the study of (Riemannian) flat solvmanifolds. A Riemannian manifold is said to be flat if its Levi-Civita (metric) connection has curvature zero. In this case, we say "M has a flat metric".

A solvmanifold is a manifold that is the homogeneous space of a connected solvable Lie group. In other words, it is the quotient S/H of a connected Lie group S by a closed subgroup H. By a flat solvmanifold, we shall mean a solvmanifold for which there is some flat Riemannian metric. Note that it is not assumed that S act on S/H by isometries (relative to the flat metric). It is, in fact, known that if M is a Riemannian flat manifold with a transitive group of isometries, then $M \approx T^m \times R^k$ for some m and k. (See Wolf [**25**, p. 88].)

Questions concerning the classification of flat manifolds and of solvmanifolds have been studied by a number of mathematicians, and many papers have been written giving various partial results. (See Bieberbach [**11**] and [**12**], Auslander [**3**], Auslander and Kuranishi [**6**], Charlap [**13**], Charlap and Vasquez [**14**], [**15**] on flat manifolds, and Chevalley [**16**], Malcev [**19**], Mostow [**21**], Auslander [**2**], Auslander and Szczarba [**7**], [**8**], [**9**] on solvmanifolds.) In [**1**], L. Auslander showed that there was a relationship between certain

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compact solvmanifolds and locally affine spaces. In [5], he and M. Auslander characterized the fundamental groups of those compact flat solvmanifolds with a discrete isotropy group. Our main compact result extends this work of Auslander and Auslander, giving a complete algebraic characterization of the fundamental groups of all compact flat solvmanifolds. The class of compact manifolds satisfying both conditions (i.e. flat solvmanifolds) is rich enough to provide a diverse collection of examples and, on the other hand, is sufficiently restricted that it has been possible to develop a straightforward constructive algorithm which generates all of these manifolds from basic algebraic data. (See §V.)

The question of classifying noncompact solvmanifolds has been treated by Mostow, who showed [21, p. 25] that any solvmanifold is regularly covered a finite number of times by the direct product of a compact solvmanifold and a Euclidean space. Auslander and Tolimieri were able [10] to prove that a solvmanifold can be realized as the total space of a vector bundle over a compact solvmanifold. Auslander and Szczarba have now established a classification scheme based on the Stiefel-Whitney classes of the vector bundles that naturally arise. (See [8].) Using this work, it is shown that the compact classification results (Theorems 1 and 3) lead directly to noncompact theorems which, for the case of flat solvmanifolds, considerably extend those of Auslander and Szczarba.

Let us consider now, in more technical detail, a summary of the results to follow. We wish to study the following two problems:

1. Classify up to diffeomorphism the compact flat solvmanifolds.

2. Given a fixed compact flat solvmanifold M, classify up to diffeomorphism the noncompact W which are total spaces of vector bundles over M. (Note the Auslander and Tolimieri result cited above.)

We approach problem 1 by first showing in Theorem 1 that the fundamental group of a compact flat solvmanifold can be completely characterized in terms of extension classes $\mu \in H_\theta^2(Z^b, Z^a)$ of finite order associated with finite integral representations of free abelian groups, $\theta: Z^b \to \mathrm{GL}(a, Z)$. (See §II, Theorems 1 and 2.) This characterization can then be used as a basis for the classification of compact flat solvmanifolds. It turns out that we need consider only a special class of representations, those without fixed points.

Using these ideas a complete listing is obtained of all compact flat solvmanifolds M of dimensions 3, 4, and 5 (except the 5-dimensional M with first betti number 1). For these computations, we use the classification of finite subgroups of GL(2, Z) and GL(3, Z) given, respectively, in Newman [22] or Seligman [23] and Tahara [24].

Further subgroup results can be used to get further solvmanifold results. However, the classification of finite abelian subgroups of GL(n, Z) is generally a hard unsolved problem in integral representation theory.

The noncompact classification question (problem 2 above) can also be expressed (in part) in terms of extension classes $\mu \in H_\theta^2(Z^b, Z^a)$ associated with integral representations of free abelian groups $\theta: Z^b \to \mathrm{GL}(a, Z)$. A complete listing of all those noncompact W over M such that the holonomy of M has odd order follows, as do other cases. (The answer when $M = T^b$ was given by Auslander and Szczarba in **[8]**. Our noncompact results derive strongly from this paper.)

Restricting to the special case of split flat solvmanifolds M (see §II for definition) leads to a very complete and simple resolution of problem 2. The case of cyclic even order holonomy is completely characterized. Also if the holonomy of compact split flat M is isomorphic to $Z_2 \times \cdots \times Z_2$, then the W over M can be described in a particularly easy manner.

This paper is divided into five sections. §II contains statements of some main theorems along with a few corollaries and examples. §§III and IV contain the proofs of the compact (respectively, noncompact) manifold theorems stated in §II, as well as the statement and proof of a number of other results. §V is a list of the compact and noncompact low dimensional flat solvmanifolds mentioned above.

The results of this paper are from part of my thesis. (This is listed as **[20]**.) I would like to thank my adviser, Robert Szczarba, for the suggestions and encouragement he has given me covering every aspect of this work. I would also like to thank R. Lee and W. Dwyer for very useful conversations on portions of this material. Finally I want to thank the referee for several suggestions and corrections.

II. Statements of main theorems and results.

1. *Characterization of the fundamental groups of compact flat solvmanifolds.* The following theorem characterizes those groups which can be the fundamental groups of flat solvmanifolds. If the action θ of Z^b on Z^a induced by some sequence

$$1 \to Z^a \to \Gamma \to Z^b \to 1$$

is such that $\theta(Z^b)$ is a finite subgroup of $\mathrm{GL}(a, Z)$, we say that "the sequence induces a finite action." Given $1 \to Z^a \to \Gamma \to Z^b \to 1$ with action $\theta: Z^b \to \mathrm{GL}(a, Z)$, there is some $\mu \in H_\theta^2(Z^b, Z^a)$ corresponding to the congruence class of this sequence. (See Mac Lane **[18**, Chapter 4], for a discussion of extensions and group cohomology.) If μ has finite order, we say that "the sequence corresponds to an extension class of finite order."

THEOREM 1. *Let Γ be a given group. Then there is a compact flat solvmanifold M such that $\Gamma \approx \pi_1(M)$ if and only if there is a short exact sequence*

$$1 \to Z^a \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} Z^b \to 1$$

such that

(a) *the sequence induces a finite action,*

(b) *the sequence corresponds to an extension class of finite order.*

This theorem can be thought of as a generalization of the following result of Auslander and Auslander [5, p. 934, Theorem C]. We say "solvmanifold M has a discrete isotropy group" if $M = S/H$, where S is a connected solvable Lie group and H is a discrete subgroup.

THEOREM 2. *Let Γ be a given group. Then there is a compact flat solvmanifold M with discrete isotropy group such that $\Gamma \approx \pi_1(M)$ if and only if there is a short exact sequence $1 \to Z^a \to^{\alpha} \Gamma \to^{\beta} Z^b \to 1$ such that*

(a) *as above,*

(b) *as above,*

(c) *θ' exists such that*

$$\begin{array}{ccc} Z^b & \xrightarrow{\theta} & \mathrm{GL}(a, Z) \\ \downarrow{\scriptstyle incl.} & & \downarrow{\scriptstyle incl.} \\ R^b & \xrightarrow{\theta'} & \mathrm{GL}(a, R) \end{array}$$

is commutative.

The klein bottle is an example of a solvmanifold that does not have discrete isotropy group. It is easy to see that (c) does not hold in this case.

We present a proof of Theorem 1 in §III. Part 1 of this proof was considerably shortened following suggestions by the referee. In Morgan [20], alternative proofs of both Theorems 1 and 2 are given which involve the explicit construction of Γ as a discrete subgroup of $E(n)$, the Euclidean motion group. This explicit proof is useful in that it sheds light on the construction of examples. (See §V.) Also, the required properties of Γ can be shown without reference to results from Auslander [1], as is required for the shorter proof given in this paper.

We denote by $M(\theta, \mu)$ the flat solvmanifold given by θ and μ.

COROLLARY. *If θ and μ are as in Theorem 1, then $M(\theta, \mu) \to M(\theta, 0)$ is a finite covering space of degree s^a where $s =$ order(μ).*

PROOF. See §III.

We call $M(\theta, 0)$ the "split flat solvmanifold" associated with $M(\theta, \mu)$.

2. *The classification of compact flat solvmanifolds.* The next theorem gives algebraic conditions which determine when $M(\theta, \mu)$ and $M(\theta', \mu')$ are diffeomorphic and (along with Theorem 1) is the basis for the classification results which follow. First we need some definitions.

DEFINITION. Let $\theta, \theta': Z^b \to \mathrm{GL}(a, Z)$ be given. We say θ' is weakly equivalent to θ, denoted $\theta' \sim \theta$, if there are $g \in \mathrm{GL}(b, Z)$ and $h \in \mathrm{GL}(a, Z)$ such that $\theta(g(t)) = h \cdot \theta'(t) \cdot h^{-1}$ for all $t \in Z^b$. If it is necessary to specify (g, h), we say "θ' is weakly equivalent to θ via (g, h)", denoted "$\theta' \sim \theta$ via (g, h)".

DEFINITION. Let θ, θ' be two given representations of Z^b in $\mathrm{GL}(a, Z)$. Let $\mu \in H^2_\theta(Z^b, Z^a)$ and $\mu' \in H^2_{\theta'}(Z^b, Z^a)$. We say μ' is weakly equivalent to μ, denoted $\mu' \sim \mu$, if $\theta' \sim \theta$ via (g, h) and $h_* \circ g^{-1*}(\mu') = \mu$. (See Mac Lane [**18**, Chapter 4].) If necessary, we use "$\mu' \sim \mu$ via (g, h)" as above.

THEOREM 3. *Let $M(\theta, \mu)$ and $M(\theta', \mu')$ be given compact flat solvmanifolds with $\theta, \theta': Z^b \to \mathrm{GL}(a, Z)$, $\mu \in H^2_\theta(Z^b, Z^a)$, and $\mu' \in H^2_{\theta'}(Z^b, Z^a)$. Suppose that b is the first betti number of both $M(\theta, \mu)$ and $M(\theta', \mu')$. Then $M(\theta, \mu)$ is diffeomorphic to $M(\theta', \mu')$ if and only if there is $(g, h) \in \mathrm{GL}(b, Z) \times \mathrm{GL}(a, Z)$ such that $\mu' \sim \mu$ via (g, h).*

We should note here that given any compact flat solvmanifold M, we can always find θ and μ such that $M \approx M(\theta, \mu)$ and $\theta: Z^b \to \mathrm{GL}(a, Z)$ with b equal to the first betti number of M.

PROOF. We know $M(\theta, \mu)$ is diffeomorphic to $M(\theta', \mu')$ if and only if $\Gamma(\theta, \mu)$ is isomorphic to $\Gamma(\theta', \mu')$. Also, we know there exists an exact sequence $1 \to Z^a \to^\alpha \Gamma \to^\beta Z^b \to 1$ with Z^b action on Z^a given by θ and extension class $\mu \in H^2_\theta(Z^b, Z^a)$. Therefore, the proof will follow if the existence of an isomorphism $f: \Gamma(\theta, \mu) \to \Gamma(\theta', \mu')$ implies the existence of the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z^a & \xrightarrow{\alpha} & \Gamma(\theta, \mu) & \xrightarrow{\beta} & Z^b & \longrightarrow & 1 \\ & & \downarrow h & & \downarrow f & & \downarrow g & & \\ 1 & \longrightarrow & Z^a & \xrightarrow{\alpha'} & \Gamma(\theta', \mu') & \xrightarrow{\beta'} & Z^b & \longrightarrow & 1 \end{array}$$

But since b is the first betti number of both manifolds, β and β' are canonically characterized as the quotient maps

$$\Gamma \to \Gamma/[\Gamma, \Gamma] = H_1(M) \to H_1(M)/\mathrm{torsion}(H_1(M)) \approx Z^b$$

where $M = M(\theta, \mu)$ or $M(\theta', \mu')$. Therefore, f naturally induces g and the existence of h follows. Q.E.D.

To carry out the classification of compact flat solvmanifolds M, we must consider the possibility that there may exist sequences

$$1 \to Z^c \to \pi_1(M) \to Z^d \to 1$$

with $d \neq b$, where b is equal to the first betti number of M. This is fully discussed in §III. Briefly, $d = b$ if and only if $\theta(Z^d) \subseteq \mathrm{GL}(c, Z)$ has no fixed points. A classification scheme is then outlined based on Theorems 1, 3, and this result. This scheme is used to classify the low dimensional cases mentioned previously. (See §V.)

3. *Basic results on the classification of noncompact flat solvmanifolds*. Given a compact flat solvmanifold M, we wish to classify (by diffeomorphism) the noncompact flat solvmanifolds W that can be realized as total spaces of vector bundles over M. (See the comment on the Auslander-Tolimieri result in §I.) In **[8]**, Auslander and Szczarba showed that the set of all such W can be put in 1-1 correspondence with a certain set $\bar{S} = \bar{S}(M)$ if a particular condition on M holds. We describe this briefly below and in more detail in §IV.

If b is equal to the first betti number of M, then $\bar{S}(M) = S/G(M)$, where $S = H^1(T^b, Z_2) \times H^2(T^b, Z_2)$, $G(M)$ is a unique subgroup of $\mathrm{SL}(b, Z_2)$ determined by M, and the action of $G(M)$ on S is the restriction of the natural action of $\mathrm{SL}(b, Z_2)$ on S.

If M is any compact solvmanifold with first betti number b, then there is a "canonical torus fibration" $\pi: M \to T^b$ with fibre a nilmanifold. (See Auslander and Szczarba **[8]**.) The condition Auslander and Szczarba need is that $\pi^*: H^2(T^b, Z_2) \to H^2(M, Z_2)$ be 1-1.

Thus the noncompact classification problem for M with π^* 1-1 breaks up into two problems:

1. Given M, find $G(M) \subseteq \mathrm{SL}(b, Z_2)$.
2. Given $G(M) \subseteq \mathrm{SL}(b, Z_2)$, determine $S/G(M)$.

The following proposition reduces the first problem above to simple algebraic conditions.

PROPOSITION 4. *The matrix* $\bar{g} \in \mathrm{SL}(b, Z_2)$ *is in* $G(M)$ *if and only if there is a* $g \in \mathrm{GL}(b, Z)$ *and* $h \in \mathrm{GL}(a, Z)$ *such that* $\mu \sim \mu$ *via* (g, h) *and* $g \equiv \bar{g}$ mod Z_2.

PROOF. See §III.

REMARK. If $M = M(\theta) = M(\theta, 0)$ is a split flat solvmanifold, the above conditions for $\bar{g} \in \mathrm{SL}(b, Z_2)$ to be in $G(M)$ reduce to conditions on θ alone.

To summarize, a compact flat solvmanifold M is defined by $\mu \in H^2_\theta(Z^b, Z^a)$ (Theorems 1 and 3). This μ also defines the noncompact flat solvmanifolds W over M if π^* is 1-1. This is because Proposition 4 indicates that $G(M)$ is determined by μ, and the Auslander-Szczarba results show that the W can be obtained from $G(M)$. In §IV, we state and prove a number of results which reduce many of the classification problems discussed above to routine matrix calculations. By way of illustrating the flavor of these results, several corollaries are presented here which can be stated with a minimum of preliminaries and notation.

COROLLARY 1 TO THEOREM 11. *If the order of the holonomy group of M is odd, then* $G(M) = \mathrm{SL}(b, Z_2)$.

COROLLARY 2 TO THEOREM 9. *If the holonomy group of* $M = M(\theta, \mu)$ *is cyclic of even order and the order of* μ *is odd, then*

$$G(M) = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \begin{matrix} \}\, b-1 \\ \}\, 1 \end{matrix} \right\} \subseteq SL(b, Z_2).$$

REMARK. Note that if $\mu = 0$, then the order of μ is odd. Therefore, this part of the hypothesis is immediate for split M.

COROLLARY 1 TO PROPOSITION 12. *If the order of* μ *is odd, then* $\pi^*\colon H^2(T^b, Z_2) \to H^2(M, Z_2)$ *is* 1-1, *where* $\pi\colon M \to T^b$ *is the canonical torus fibration for* $M = M(\theta, \mu)$.

This corollary gives a sufficient condition for the Auslander-Szczarba hypothesis mentioned above. Also, it implied that π^* is 1-1 if the holonomy group $\theta(Z^b)$ has odd order. This is because μ has odd order if $\theta(Z^b)$ has odd order.

PROPOSITION (FROM MATERIAL IN §IV.4). 1. *If the holonomy group of* $M = M(\theta, \mu)$ *has odd order, then there are* $3(b/2) + 1$ *noncompact solvmanifolds over M if b is even and* $3((b-1)/2) + 2$ *noncompact solvmanifolds over M if b is odd.*

2. *If the holonomy group of M is cyclic of even order and* μ *has odd order, then there are* $(11b - 10)/2$ *noncompact solvmanifolds over M if b is even and* $(11(b-1)/2) + 2$ *noncompact solvmanifolds over M if b is odd.*

III. Compact theorems.

1. *Proof of Theorem* 1.

Part 1. Given $1 \to Z^a \to \Gamma \to Z^b \to 1$ with (a) and (b) holding, we wish to show that $\Gamma \cong \pi_1(M)$, where M is some compact flat solvmanifold.

By Theorem C, p. 964, of Auslander [1], it suffices to show that Γ is a torsion free Bieberbach group.

Clearly Γ is torsion free. Let $K = \ker(\theta)$. Then $H^2(K, Z^b)$ is free. Let $f\colon K \to Z^b$. Since μ has finite order, $f^*(\mu) = 0$. Hence the extension A of K by Z^a corresponding to $f^*(\mu)$ is free abelian. But $\Gamma/A = \Gamma/Z^a/A/Z^a = Z^b/K = \theta(Z^b)$.

Part 2. Let $\Gamma = \pi_1(M)$ where M is a flat solvmanifold. We wish to show that there is a short exact sequence $1 \to Z^a \to^{\alpha} \Gamma \to^{\beta} Z^b \to 1$ such that (a) and (b) hold.

Since M is a compact solvmanifold, there is a short exact sequence $1 \to N \to^{\alpha} \Gamma \to^{\beta} Z^b \to 1$ where N is nilpotent. See Mostow [**21**, p. 27]. Since M is flat, it follows that Γ is torsion free and there is a maximal normal free abelian subgroup $K \subseteq \Gamma$ of finite index. Therefore N contains a free abelian normal subgroup of finite index, $N' = N \cap K$.

Lemma 1. *If N is a torsion free nilpotent group and $N' \subseteq N$ is an abelian normal subgroup of N of finite index, then N is free abelian.*

Proof. This lemma follows from an easy induction on the length of the upper central series of N, using the result of Malcev [**19**, p. 298, Corollary 2] that N/Z is torsion free where Z is the center of N.

By Lemma 1, $N \approx Z^a$, and we have the short exact sequence $1 \to Z^a \to^{\alpha} \Gamma \to^{\beta} Z^b \to 1$. Let $K \subseteq \Gamma$ be the maximal normal free abelian subgroup of Γ of finite index.

Lemma 2. $K \subseteq \alpha(Z^a)$.

Proof. Let $K' = \alpha(Z^a) \cap K$. Then $K' \subseteq \alpha(Z^a)$ is a subgroup of finite index. Let $x \in \alpha(Z^a)$. Define

$$\sigma_x: \alpha(Z^a) \to \alpha(Z^a) \quad \text{by } \sigma_x(y) = x^{-1}yx.$$

Clearly $\sigma_x|K' =$ identity. Then $\sigma_x|K =$ identity, by the following

Lemma 3. *Let A be a finitely generated free abelian group and let A' be a subgroup of finite index. Let σ be an endomorphism of A which is the identity on A'. Then σ is the identity on A.*

This lemma is found in Auslander and Auslander [**5**, p. 937]. Therefore, it follows that any $x \in \alpha(Z^a)$ commutes with every $y \in K$. Let $K^0 = K \cdot \alpha(Z^a)$. Since both K and $\alpha(Z^a)$ are normal subgroups of Γ, K^0 is a normal subgroup. The observations above show that K^0 is abelian. Therefore, by the maximality of K, we get $K = K^0$. Therefore, $\alpha(Z^a) \subseteq K$. This completes the proof of Lemma 2.

Consider now the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z^a & \xrightarrow{\alpha'} & Z^n & \xrightarrow{\beta'} & Z^b & \longrightarrow & 1\\
 & & \approx\downarrow & & \downarrow\sigma & & \downarrow\sigma' & & \\
1 & \longrightarrow & Z^a & \xrightarrow{\alpha} & \Gamma & \xrightarrow{\beta} & Z^b & \longrightarrow & 1\\
 & & \downarrow & & \downarrow\delta & & \downarrow\delta' & & \\
 & & 1 & \longrightarrow & \Delta & \xrightarrow[\approx]{} & \Delta & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Note that $K = \sigma(Z^n)$. It easily follows that the action of Z^b on Z^a can be identified with the (holonomy) action of Δ on Z^a. Therefore, we have established (a) of Theorem 1.

We now wish to show that (b) holds. This is immediate from Auslander and Auslander [5, Theorem 2.2, p. 938]. This completes the proof of Theorem 1.

3. *Proof of the Corollary to Theorem* 1. Let $h: Z^a \to Z^b$ be multiplication by s. Then $H^2_\theta(Z^b, Z^a) \to^{h_*} H^2_\theta(Z^b, Z^a)$ carries $\mu \mapsto 0$, since $\operatorname{order}(\mu) = s$. Thus we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z^a & \longrightarrow & \Gamma(\theta,\mu) & \longrightarrow & Z^b & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & Z^a & \longrightarrow & \Gamma(\theta) & \longrightarrow & Z^b & \longrightarrow & 1 \end{array}$$

where the second sequence is split, and f is 1-1 since h is 1-1. Thus $\Gamma(\theta)$ contains $\Gamma(\theta, \mu)$ as a subgroup of index s^a. Q.E.D.

4. We must consider the following problem before discussing the classification of compact flat solvmanifolds. Let $\Gamma = \pi_1(M)$ for some M. Then we know there is a short exact sequence $1 \to Z^a \to \Gamma \to Z^b \to 1$ with b the first betti number of M, and this sequence is essentially unique by Theorem 3. However, it will, in general, happen that there are sequences $1 \to Z^c \to \Gamma \to Z^d \to 1$ with d not equal to the first betti number of M, and there may be several nonisomorphic sequences for the same d. (It can be shown, however, that d must be between b and r where r is the rank of the holonomy group of Γ.) The following example illustrates these possibilities.

EXAMPLE. We define a four dimensional M with first betti number 3. Let $\Gamma = \{v, u_1, u_2, u_3\}$ where

$$v = \left[\begin{array}{c|c} & 2 \\ & 0 \\ I & 0 \\ & 0 \\ \hline 0 & 1 \end{array}\right], \qquad u_1 = \left[\begin{array}{c|c} & 1 \\ & 1 \\ I & 0 \\ & 0 \\ \hline 0 & 1 \end{array}\right],$$

$$u_2 = \left[\begin{array}{c|c} & 0 \\ & 0 \\ I & 1 \\ & 0 \\ \hline 0 & 1 \end{array}\right], \qquad u_3 = \left[\begin{array}{cccc|c} -1 & & & & 0 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 1 \\ \hline & & 0 & & 1 \end{array}\right].$$

Then

$$\begin{array}{llll} 1 \to & Z \to & \Gamma \to Z^3 \to 1 \\ & e \mapsto & v \\ & & u_i \mapsto t_i \end{array}$$

and Γ has first betti number 3. However,

$$\begin{array}{llll} 1 \to & Z^2 \to & \Gamma \to Z^2 \to 1 \\ & e_1 \mapsto & v \\ & e_2 \mapsto & u_1 \\ & & u_2 \mapsto t_1 \\ & & u_3 \mapsto t_2 \end{array}$$

and

$$\begin{array}{llll} 1 \to & Z^2 \to & \Gamma \to Z^2 \to 1 \\ & e_1 \mapsto & v \\ & e_2 \mapsto & u_2 \\ & & u_1 \mapsto t_1 \\ & & u_3 \mapsto t_2 \end{array}$$

Since the first sequence is split and the second is not, these cannot be isomorphic as sequences.

The following theorem gives a simple test for picking out "redundant" sequences and will prove very useful in the compact classification scheme described below.

DEFINITION. Let θ: $Z^d \to \mathrm{GL}(c, Z)$. Then θ (or $\theta(Z^d)$) has a fixed point $x \in Z^c$ if $Ax = x$ for all $A \in \theta(Z^d)$.

THEOREM 5. *Let* θ: $Z^d \to \mathrm{GL}(c, Z)$ *be given. Then* $M(\theta, \mu)$ *has first betti number d if and only if* θ *has no nonzero fixed point* (*any* μ).

PROOF. From the low dimensional Serre spectral sequence for the fibration $T^c \to M \to T^d$, we get

$$0 \to H^1(T^d, Q) \to H^1(M, Q)$$
$$\to H^0(Z^d, H^1(Z^c)) \otimes Q \xrightarrow{d_2} 0$$

because μ has finite order implies d_2 has finite order (Z coefficients). But $H^0(Z^d, H^1(Z^c)) \otimes Q$ is equal to the fixed points of the Z^d action on $H^1(Z^c, Q)$. This can be identified with the fixed points of the Z^d action on $H_1(Z^c, Q)$ because θ is a finite action. Q.E.D.

Now consider the following scheme for classifying all compact flat solvmanifolds of dimension n.

(a) For all a, b such that $a + b = n$, determine the (g, h)-equivalence classes of finite representations $\theta\colon Z^b \to \mathrm{GL}(a, Z)$ without nonzero fixed points, $(g, h) \in \mathrm{GL}(b, Z) \times \mathrm{GL}(a, Z)$.

This could be accomplished, say, by listing the conjugacy classes of finite abelian subgroups of $\mathrm{GL}(a, Z)$ for all $a < n$, eliminating those classes G with nonzero fixed points, and then taking the remaining classes and reducing them by (g, h)-equivalence.

It should be noted, however, that, in general, the problem of finding the conjugacy classes of finite abelian subgroups of $\mathrm{GL}(a, Z)$ is a hard unsolved problem in group representation theory. Some special cases have been solved: cyclic subgroups of prime order (Curtis and Reiner [17]), $a = 2$ (Newman [22] and Seligman [23]), and $a = 3$ (Tahara [24]).

(b) For each representation $\theta\colon Z^b \to \mathrm{GL}(a, Z)$ in the list resulting from (a) above, compute torsion$(H^2_\theta(Z^b, Z^a))$ and determine the (g, h)-equivalence classes of the $\mu \in$ torsion$(H^2_\theta(Z^b, Z^a))$.

Now by Theorems 1, 3, and 5, the list of μ obtained from (b) gives the diffeomorphism classes of all compact flat n-dimensional solvmanifolds.

A classification of low dimensional flat compact solvmanifolds has been carried out using the scheme outlined above. See §V.

We would now like to make some comments on compact split flat solvmanifolds. Given a finite representation $\theta\colon Z^b \to \mathrm{GL}(a, Z)$ without fixed points, there is at least one compact flat solvmanifold $M(\theta, \mu)$, namely the split one ($\mu = 0$), and perhaps no other, if torsion$(H^2_\theta(Z^b, Z^a)) = 0$. Therefore, the split flat solvmanifolds form a large class of flat solvmanifolds. Their classification is essentially the same as the classification of (g, h)-equivalence classes of finite representations $\theta\colon Z^b \to \mathrm{GL}(a, Z)$ without fixed points.

IV. Noncompact theorems.

1. *Basic noncompact theorems.* Given a compact flat solvmanifold M, we wish to classify (by diffeomorphism) the noncompact flat solvmanifolds W that are total spaces of vector bundles over M.

THEOREM 6 (AUSLANDER-SZCZARBA). *Suppose $E \to M$ is a vector bundle. Then E is a solvmanifold if and only if E is the pullback of a sum of line bundles over T^b via the canonical torus fibration for M. In other words, if and only if there is a vector bundle $E' \to T^b$ where E' is a sum of line bundles such that*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ \pi\ } & T^b \end{array}$$

commutes.

THEOREM 7 (AUSLANDER-SZCZARBA). *Let $\pi\colon M \to T^b$ be the canonical torus* *fibration for M. Suppose* $\pi^*: H^2(T^b, Z_2) \to H^2(M, Z_2)$ *is* 1-1. *Then* W_1 *and* W_2 *over M are diffeomorphic if and only if there is some* $f \in \mathrm{Diff}(M)$ *such that* $f^*: H^*(M, Z_2) \to H^*(M, Z_2)$ *carries the first and second Stiefel-Whitney classes of* $W_1 \to M$ *to those of* $W_2 \to M$.

The above two theorems are actually valid for all (not necessarily flat) solvmanifolds and are proven in this generality by Auslander and Szczarba. It is easy to find compact flat solvmanifolds M for which π^* is not 1-1. Proposition 12, stated below, gives a characterization of those $M(\theta, \mu)$ with π^* 1-1.

Now suppose we have M with π^* 1-1. Given $(w_1, w_2) \in H^1(T^b, Z_2) \times H^2(T^b, Z_2)$, it is easy to construct a sum of line bundles $E' \to T^b$ such that its first two Stiefel-Whitney classes are (w_1, w_2). Let

$$S = H^1(T^b, Z_2) \times H^2(T^b, Z_2).$$

Since $\pi^*: H^k(T^b, Z_2) \to H^k(M, Z_2)$ is injective for $k = 1, 2$, we may consider $S \subseteq H^1(M, Z_2) \times H^2(M, Z_2)$. Then $\mathrm{Diff}(M)$ acts on S, and by Theorems 6 and 7 the solvmanifolds over M may be identified with the orbit space $S/\mathrm{Diff}(M)$. We should observe that if $W^m, \overline{W}^{\overline{m}} \to M$ are two bundles over M with dimensions m and $\overline{m}$, respectively, $m < \overline{m}$, and $(w_1(W), w_2(W)) = (w_1(\overline{W}), w_2(\overline{W}))$, then $\overline{W} = W \times R^t$ where $t = \overline{m} - m$. Then the previous comment about $S/\mathrm{Diff}(M)$ refers to solvmanifolds of "minimal dimension".

Observe that any $f \in \mathrm{Diff}(M)$ induces a $g \in \mathrm{Diff}(T^b)$ by projecting via the canonical torus fibration. Therefore, we have a map from $\mathrm{Diff}(M)$ to $\mathrm{Diff}(T^b)$. We say $g \in \mathrm{Diff}(T^b)$ "lifts to $\mathrm{Diff}(M)$" if it is in the image of this map. Let

$$H = \{ g \in \mathrm{Diff}(T^b) | g \text{ lifts to } \mathrm{Diff}(M) \}.$$

Then clearly $S/\mathrm{Diff}(M) = S/H$ with the obvious actions.

Also observe that since S consists of pairs of Z_2 classes, the $\mathrm{Diff}(T^b) = \mathrm{GL}(b, Z)$ action on S factors through a $\mathrm{SL}(b, Z_2)$ action. Therefore, if $\overline{H}$ is the Z_2 reduction of the elements in H, then $S/\mathrm{Diff}(M) = S/H = S/\overline{H}$.

DEFINITION. $\overline{H}$, defined as above for M, is denoted $G(M)$. Thus $G(M) \subseteq \mathrm{SL}(b, Z_2)$ and completely determines the noncompact solvmanifolds over M, assuming π^* is 1-1.

REMARK. Frequently we will think of $G(M)$ as a group of matrices. This requires a choice of basis, which will always be made as follows. Choose $t_1, \ldots, t_b \in Z^b$ where $\theta(t_i) = B_i$ and the order of B_i divides the order of B_{i+1} for all i.

Then

$$\theta(Z^b) = \{B_1, \ldots, B_b\} \cong Z_{m_1} \times \cdots \times Z_{m_b}$$

where $m_i = \mathrm{order}(B_i)$. Note that $m_i = 1$ is possible. We take $g \in \mathrm{SL}(b, Z_2)$

relative to the basis $t_1, \ldots, t_b$ where the ith column of g gives $g(t_i)$.

The following theorem helps interpret the Auslander-Szczarba theorems in terms of flat solvmanifolds.

THEOREM 8. *Let $M = M(\theta, \mu)$ be given where θ: $Z^b \to \mathrm{GL}(a, Z)$ is a finite representation and $\mu \in H_\theta^2(Z^b, Z^a)$ has finite order, as usual. We assume that b is the first betti number of M. Suppose $g \in \mathrm{GL}(b, Z)$. Then g lifts to* Diff(M) *if and only if there is an $h \in \mathrm{GL}(a, Z)$ such that $\mu \sim \mu$ via (g, h).*

PROOF. This follows from the proof of Theorem 3 and the definitions.

Proposition 4 follows at once.

We noted in §II.3 our interest in the problem: given $M = M(\theta, \mu)$, find $G(M) \subseteq \mathrm{SL}(b, Z_2)$. It is convenient to consider first the special case in which $\mu = 0$. (Recall that $M(\theta, \mu)$ with $\mu = 0$ is said to be "split".) This is done in subsection 2 below, followed in 3 by a discussion of the general case. We consider in 4 the other basic problem described in §II.3, that of finding $S/G(M)$. Subsections 5–7 contain an important characterization of the condition "π^* is 1-1" and the proofs of the theorems stated in 2–4.

2. *Finding $G(M)$ for split flat solvmanifolds.* We wish to consider $M = M(\theta, \mu)$ with $\mu = 0$ and θ: $Z^b \to \mathrm{GL}(a, Z)$ where b is equal to the first betti number of M. In this subsection we will show that $G(M)$ factors into two parts determined by the "odd and even summands" of the holonomy of M and that the part of $G(M)$ determined by the odd summand of the holonomy is completely known (Theorem 9). We are also able to characterize the even part of $G(M)$.

REMARK. The Auslander-Szczarba condition "π^* is 1-1" is always satisfied for split flat solvmanifolds. See 5 below.

NOTATION. $M = M(\theta)$ denotes the split flat solvmanifolds $M(\theta, 0)$.

THEOREM 9. *Let $M = M(\theta)$ be a given compact split flat solvmanifold. Let $t_1, \ldots, t_b$ be a basis for Z^b such that* order$(\theta(t_i))$ *is odd for $i = 1, \ldots, s$ and* order$(\theta(t_i))$ *is even for $i = s + 1, \ldots, b$. Now define θ': $Z^{b-s} \to \mathrm{GL}(a, Z)$ by letting $t_{s+1}, \ldots, t_b$ generate Z^{b-s} and setting $\theta'(t_i) = \theta(t_i)$ for $i = s + 1, \ldots, b$. Then*

$$G(M) = \left\{ \begin{array}{|c|c|} \hline * & * \\ \hline 0 & Z \\ \hline \end{array} \in SL(b, Z_2) \mid Z \in G(M(\theta')) \right\}.$$

REMARK. Note that $\theta(t_i) = I$ (order 1) is not excluded.

The proof of Theorem 9 is given in subsection 6.

COROLLARY 1. *If $\theta(Z^b)$ has odd order, then $G(M) = \mathrm{SL}(b, Z_2)$.*

Thus, using this corollary and the material in §II.3, we see that the noncompact split flat solvmanifolds over M are in 1-1 correspondence with $S/\mathrm{SL}(b, Z_2)$ where $S = H^1(T^b, Z_2) \times H^2(T^2, Z_2)$. This set is completely described in subsection 4.

COROLLARY 2. *If* $s = b - 1$ (*i.e., if* $\theta(Z^b) \cong Z_{m_1} \times \cdots \times Z_{m_b}$ *where* $m_1, \ldots, m_{b-1}$ *are odd and* m_b *is even*), *then*

$$G(M) = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \begin{matrix} \}\ b-1 \\ \}\ 1 \end{matrix} \right\} \subseteq SL(b, Z_2).$$

Combining this result with §II.3, it follows that the noncompact split flat solvmanifolds over M are in 1-1 correspondence with $S/G(M)$. This set is completely described in subsection 4.

REMARK. In particular, Corollary 2 applies when the holonomy group of M, $\theta(Z^b)$, is cyclic of even order.

The corollaries follow at once from Theorem 9.

Thus Theorem 9 shows that $G(M)$ breaks up into an "odd part" and an "even part". Further, Theorem 9 completely determines the odd part of $G(M)$. Now, in Theorem 10 below, the even part is given a simple categorization. First, however, we need some preliminaries.

If $K \subseteq \mathrm{GL}(a, Z)$, let $N(K)$ denote the normalizer of K in $\mathrm{GL}(a, Z)$ and $C(K)$ the centralizer of K in $\mathrm{GL}(a, Z)$. If K is abelian and has rank b, we can define $\psi: N(K)/C(K) \to \mathrm{SL}(b, Z_2)$ to be a restriction of the natural epimorphism

$$\mathrm{Aut}(K) \to \mathrm{Aut}(K \otimes Z_2) \subseteq \mathrm{SL}(b, Z_2),$$

identifying $N(K)/C(K)$ with a subgroup of Aut(K). Note that $\mathrm{Aut}(K \otimes Z_2) = \mathrm{SL}(b, Z_2)$ if and only if $K \cong Z_{e_1} \times \cdots \times Z_{e_b}$ where $e_i | e_{i+1}$ and the e_i are even.

THEOREM 10. *Let* $M = M(\theta')$ *be given with* $\theta': Z^b \to \mathrm{GL}(a, Z)$, *and suppose that* $\theta'(Z^b) \cong Z_{e_1} \times \cdots \times Z_{e_b}$ *where* $e_i | e_{i+1}$ *and the* e_i *are even. Then there is a canonically defined subgroup* $H(\theta'(Z^b)) \subseteq N(\theta'(Z^b))/C(\theta'(Z^b))$ *such that* $G(M) = \psi(H(\theta'(Z^b)))$.

Furthermore, if $\theta(Z^b)$ *consists only of elements of order* 1, 2, 3, 4, *or* 6, *then* $H(\theta'(Z^b)) = N(\theta'(Z^b))/C(\theta'(Z^b))$ *and, therefore,* $G(M) = \psi(N(\theta'(Z^b))/C(\theta'(Z^b)))$.

Finally, if $\theta'(Z^b) \cong Z_2 \times \cdots \times Z_2 \cong Z_2^b$, *then* ψ *is an isomorphism and* $G(M) \approx N(\theta'(Z^b))/C(\theta'(Z^b))$.

Combined with Theorem 9, Theorem 10 gives a simple algebraic characterization of the way in which θ determines $G(M(\theta))$, especially in the special cases mentioned above. The proof of this theorem is given in 6 below.

Examples G_4, G_5, and G_6 in §V show that ψ need not be an isomorphism. In these cases

$$H(\theta'(Z^b)) = N(\theta'(Z^b))/C(\theta'(Z^b)) \neq \{I\},$$

but $G(M) = \{I\}$.

EXAMPLE TO ILLUSTRATE THEOREM 10. Define a split $T^3 \to M \to T^2$ with holonomy isomorphic to $Z_2 \times Z_2$ as follows. Let θ: $Z \times Z \to \mathrm{GL}(3, Z)$ be such that

$$\theta(Z^2) = K = \{\theta(1,0) = B_1, \theta(0,1) = B_2, B_1B_2, I\}$$

$$= \left\{ \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right\}.$$

Theorem 10 implies $G(M) \approx N(K)/C(K)$. It is easy to see that

$$G(M) = N(K)/C(K) = \left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then

$$S/G(M) = H^1(T^3, Z_2) \times H^2(T^3, Z_2)/G(M)$$
$$= \{(0,0), (x_1, 0), (x_2, 0), (0, x_1x_2), (x_1, x_1x_2), (x_2, x_1x_2)\}$$

where $H^1(T^3, Z_2) = \{x_1, x_2, x_3\}$.

3. *Finding $G(M)$ for nonsplit flat solvmanifolds*. We may view the results above on split flat solvmanifolds as giving necessary conditions in the general (i.e. nonsplit) case. In other words, given $M(\theta, \mu)$ we have the associated split $M(\theta)$. Then it is clear that $G(M(\theta, \mu)) \subseteq G(M(\theta))$ by Proposition 4. Thus $G(M(\theta))$ gives information about $G(M(\theta, \mu))$ but does not determine it. We should note here that $M(\theta)$ and $M(\theta, \mu)$ have the same first betti number.

However, the split result concerning odd order holonomy (Theorem 9, Corollary 1) generalizes directly to the nonsplit case. Also, the even order cyclic holonomy result (Theorem 9, Corollary 2) generalizes to the nonsplit case whenever the order of μ is odd. (See Corollaries 1 and 2 below.) Also see Proposition 12 in subsection 5.

THEOREM 11. *Let $M = M(\theta, \mu)$ where $\mu \in H^2_\theta(Z^b, Z^a)$ has odd order. Then, choosing $t_1 \cdots t_s, t_{s+1} \cdots t_b$ as in Theorem 9,*

$$G(M) \supseteq \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & I \end{array} \right) \begin{array}{l} \} s \\ \} b-s \end{array} \right\}.$$

The proof of this theorem is in subsection 7. Compare this result with Theorem 9.

COROLLARY 1. *If the order of the holonomy group of M is odd, then* $G(M) = \mathrm{SL}(b, Z_2)$.

Compare this with Corollary 1 to Theorem 9.

PROOF. If the order of the holonomy group is odd, then the order of μ is odd.

REMARK. Together with the following proposition and the material in §II.3, this corollary implies that the noncompact W over M, where the holonomy of M is odd, are given by $S/\mathrm{SL}(b, Z_2)$. Thus the W over such M can be identified with the W over T^b, where b is the first betti number of M.

COROLLARY 2. *If μ has odd order and $s = b - 1$ (i.e. if $\theta(Z^b) \cong Z_{m_1} \times \cdots \times Z_{m_b}$ where $m_1, \ldots, m_{b-1}$ are odd and m_b is even), then*

$$G(M) = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array} \right) \begin{array}{l} \} b-1 \\ \} 1 \end{array} \right\} \subseteq SL(b, Z_2).$$

Compare this with Corollary 2 to Theorem 9.

PROOF. This follows from Theorem 11 and Corollary 2 to Theorem 9, along with the first paragraph of this subsection.

REMARK. Neither Theorem 9 nor its Corollary 2 generalize directly to the nonsplit case. (See Morgan [20, pp. 65–68], for an example.)

4. *Finding $S/G(M)$*. In §II.3, we indicated our interest in the following question: Given $G(M) \subseteq \mathrm{SL}(b, Z_2)$, compute $S/G(M)$ where $S = H^1(T^b, Z_2) \times H^2(T^b, Z_2)$. The following two explicit calculations of $S/G(M)$ have been carried out:

1. $G(M) = \mathrm{SL}(b, Z_2)$. This was done by Auslander and Szczarba [8].

2.

$$G(M) = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array} \right) \begin{array}{l} \} b-1 \\ \} 1 \end{array} \right\} \subseteq SL(b, Z_2).$$

In the first case, $S/G(M)$ has $3(b/2) + 1$ classes if b is even and $3((b - 1)/2) + 2$ classes if b is odd. In the second case, $S/G(M)$ has $(11b - 10)/2$ classes if b is even and $(11(b - 1))/2$ classes if b is odd. Full details are given in Morgan [**20**].

COROLLARY. *If $M = T^b$ or if the holonomy group of $M = M(\theta, \mu)$ has odd order, then there are $3(b/2) + 1$ solvmanifolds over M if b is even and $3((b - 1)/2) + 2$ solvmanifolds over M if b is odd.*

COROLLARY. *If μ has odd order and the holonomy group of $M = M(\theta, \mu)$ is isomorphic to $Z_{m_1} \times \cdots \times Z_{m_b}$ where $m_1, \ldots, m_{b-1}$ are odd and m_b is even, then there are $(11b - 10)/2$ solvmanifolds over M if b is even and $(11(b - 1)/2) + 2$ solvmanifolds over M if b is odd.*

If M is split, then $\mu = 0$ has odd order. Also, note that this corollary includes the case that M has even order cyclic holonomy group.

These corollaries follow from the corollaries to Theorems 9 and 11, along with the above comments. More than just the number of noncompact solvmanifolds is known. Explicit representatives for each diffeomorphism class have been obtained. See Morgan [**20**, §VI].

5. *An important technical result.* Theorem 7 has as part of its hypothesis that $\pi^*: H^2(T^b, Z_2) \to H^2(M, Z_2)$ be 1-1 where $\pi: M \to T^b$ is the canonical torus fibration for M. We would like to show that this condition can be characterized directly, in a manner very useful in constructing examples.

Let $M = M(\theta, \mu)$ be a compact flat solvmanifold where $\theta: Z^b \to \mathrm{GL}(a, Z)$ and $\mu \in H_\theta(Z^b, Z^a)$. Define

$$J_\theta = \{x - \theta(t)x | x \in Z^a, t \in Z^b\}.$$

Let $\bar{J}_\theta = J_\theta \otimes Z_2$. We may identify μ with (y_{rt}), a $b(b - 1)/2$-tuple of elements of Z^a indexed by (r, t) where $1 \leq r < t \leq b$, modulo certain equivalences.

(See Mac Lane [**18**, p. 110] or Morgan [**20**, p. 23].) Let $\bar{y}_{rt} \in Z_2^a$ denote y_{rt} reduced mod 2.

PROPOSITION 12. *π^* is 1-1 if and only if $\bar{y}_{rt} \in \bar{J}_\theta$ for each (r, t).*

The proof of this proposition follows the statement and proof of several corollaries.

COROLLARY 1. *If the order of μ is odd, then π^* is* 1-1.

COROLLARY 2. *If the order of $\theta(Z^b)$, the holonomy group, is odd, then π^* is* 1-1.

PROOF. The order of μ divides the order of $\theta(Z^b)$.

DEFINITION. Given $\theta: Z^b \to \mathrm{GL}(a, Z)$, let

$$AS(\theta) = \{\mu \in \text{torsion}(H_\theta) | \pi^* \text{ for } M(\theta, \mu) \text{ is 1-1}\}.$$

Then, $\mu \in AS(\theta)$ if and only if the Auslander-Szczarba noncompact classification theorem (Theorem 7) applies to $M(\theta, \mu)$.

COROLLARY 3. *Let* $b = 2$. *If* torsion$(H_\theta) \cong Z_2^k$ *for some* k, *then* $AS(\theta) = \{\bar{0}\}$.

PROOF. If $b = 2$, then $\bar{J}_\theta = \bar{K} = K \otimes Z_2$ where $H_\theta^2(Z^2, Z^a) \approx H_\theta = Z^a/K$. Then $\mu = [(y_{12})] = \bar{0}$ if and only if $y_{12} \in K$. But $[y_{12}] \in$ torsion(H_θ) $\cong Z_2^k$. Thus $y_{12} \in K$ if and only if $\bar{y}_{12} \in \bar{K}$. Q.E.D.

COROLLARY 4. *If the holonomy group is cyclic and* torsion$(H_\theta) \cong Z_2^k$ *for some* k, *then* $AS(\theta) = \{\bar{0}\}$.

PROOF. Let θ': $Z^2 \to \mathrm{GL}(a, Z)$ have the same (cyclic) image as θ. Then it is not hard to see that $H_\theta = (Z^a)^m \times (H_{\theta'})^{b-1}$ where $m = [b(b-1)/2] - [b - 1]$. Thus torsion$(H_\theta) =$ torsion$(H_{\theta'})^{b-1} \cong Z_2^k$. But Corollary 3 above applies to $H_{\theta'}$, and the result follows. Q.E.D.

The examples in §V show that, in general, π^* is not 1-1 even in the cyclic holonomy case. In particular, if the holonomy of M is isomorphic to Z_2, then π^* is 1-1 if and only if M is split (i.e. $\mu = 0$). This follows from Corollary 4 above.

PROOF OF PROPOSITION 12. Let $\Gamma = \pi_1(M)$. We have $1 \to Z^a \to^\alpha \Gamma \to^\beta Z^b \to 1$ as usual. Then π^* is 1-1 if and only if π_*: $H_2(M, Z_2) \to H_2(T^b, Z_2)$ is onto. The homology spectral sequence for the fibration $T^a \to M \to T^b$ yields an exact sequence in low dimensions:

$$H_2(M, Z_2) \xrightarrow{\pi_*} H_2(T^b, Z_2) \xrightarrow{d_2} H_0^\theta(T^b, H_1(T^a, Z_2))$$
$$\to H_1(M, Z_2) \to H_1(T^b, Z_2) \to 0$$

where d_2 is the transgression $E_2^{2,0} \to E_2^{0,1}$. Thus π_* is onto if and only if $d_2 = 0$. Let $\mu \in H_\theta^2(Z^b, Z^a)$ be the extension class corresponding to $1 \to Z^a \to \Gamma \to Z^b \to 1$. Then d_2 can be identified with the image of μ under the composite

$$\psi\colon H_\theta^2(T^b, Z^a) \to H^2(T^b, H_0^\theta(T^b, Z^a \times Z_2))$$
$$\underset{\approx}{\to} \mathrm{Hom}(H_2(T^b, Z_2), H_0^\theta(Z^b, Z^a \times Z_2)).$$

Identify $H_\theta^2(Z^b, Z^a)$ with a subgroup of $H_\theta = (Z^a)^{b(b-1)/2}/K$ as usual, and identify $H^2(T^b)$ with $\Lambda^2(H_1(T^b)) = \Lambda^2(\pi_1(T^b)) = \Lambda^2(x_1, \ldots, x_b)$. Note that $H_0^\theta(T^b, Z^a \times Z_2) = Z_2^a/\bar{J}$. Then we may identify the composite map ψ as

$$\psi([y_{rt}])(x_i \wedge x_j) = [y_{ij}] \in Z_2^a/\bar{J}$$

where $[y_{rt}] \in H_\theta$ and $x_i \wedge x_j \in H_2(T^b)$. The result follows.

6. *The proofs of Theorems* 9 *and* 10. We write Z^b and Γ multiplicatively and Z^a additively. Thus θ: $Z^b \to \mathrm{GL}(a, Z)$ is a homomorphism from a multiplicative group to a multiplicative group.

Given $1 \to Z^a \to \Gamma \to Z^b \to 1$, we have defined what "$g \in \mathrm{GL}(b, Z)$ lifts to $\mathrm{Aut}(\Gamma)$" means in §II.3. If $g' \in \mathrm{SL}(b, Z_2)$, let "$g'$ lifts to $\mathrm{Aut}(\Gamma)$" mean that there is some $g \in \mathrm{GL}(b, Z)$ such that g lifts to $\mathrm{Aut}(\Gamma)$ and $g' \equiv g \bmod Z_2$. Note that

$$G(M) = \{ g' \in \mathrm{SL}(b, Z_2) | g' \text{ lifts to } \mathrm{Aut}(\Gamma) \approx \mathrm{Diff}(M) \}.$$

Recall that $\mathrm{SL}(b, Z_2)$ is generated by matrices $e_{ij} = I + E_{ij}$ where

$$\begin{aligned} (E_{ij})_{rt} &= 0 \quad \text{if } (r, t) \neq (i, j), \\ &= 1 \quad \text{if } (r, t) = (i, j). \end{aligned}$$

LEMMA 13. *Let* $1 \to Z^a \to \Gamma \to Z^b \to 1$ *be a given split sequence with* θ: $Z^b \to \mathrm{GL}(a, Z)$ *the induced action. Let* $\theta(Z^b) \approx Z_{m_1} \times \cdots \times Z_{m_b}$ *where* $m_i | m_{i+1}$ *for all* i. (*Note that* $m_i = 1$ *is not excluded.*) *We may suppose there is a basis* $t_1, \ldots, t_b$ *for* Z^b *such that* $\theta(t_i)$ *corresponds to a generator for* Z_{m_i}. *Then if* m_i *is odd,* $e_{ij} \in \mathrm{SL}(b, Z_2)$ *lifts to* $\mathrm{Aut}(\Gamma)$.

LEMMA 14. *Assume the hypothesis of Lemma* 13. *Suppose* e_{ij} *lifts to* $\mathrm{Aut}(\Gamma)$. *If* $2^n | m_i$, *then* $2^n | m_j$.

COROLLARY. *Assume the hypothesis of Lemma* 14. *If* e_{ij} *lifts to* $\mathrm{Aut}(\Gamma)$, *then* m_i *even implies* m_j *even.*

The proofs of Lemmas 13 and 14 are simple direct arguments. (See Morgan [**20**, pp. 54–55].)

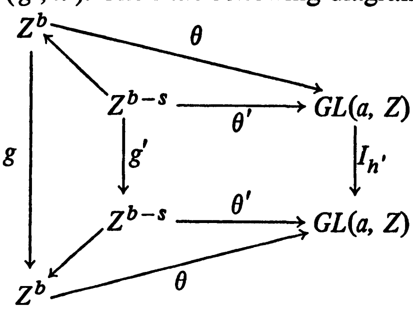
PROOF OF THEOREM 9. First observe that if $z \in G(M(\theta'))$, then

$$\begin{bmatrix} I & 0 \\ 0 & z \end{bmatrix} \in G(M(\theta)).$$

This can be seen as follows. Let $g' = z$, and let

$$g = \begin{bmatrix} I & 0 \\ 0 & z \end{bmatrix}.$$

We have $\theta' \sim \theta'$ via (g', h'). Then the following diagram commutes:



Similarly, if

$$\boxed{\begin{matrix} I & 0 \\ 0 & z \end{matrix}} \in G(M)$$

then $z \in G(M(\theta'))$. Lemma 13 implies that any

$$\boxed{\begin{matrix} x & y \\ 0 & I \end{matrix}} \in G(M),$$

but

$$\boxed{\begin{matrix} x & y \\ 0 & z \end{matrix}} = \boxed{\begin{matrix} I & 0 \\ 0 & z \end{matrix}}\boxed{\begin{matrix} x & y \\ 0 & I \end{matrix}}.$$

By the corollary to Lemma 14, we know no other g can lift. This completes the proof of Theorem 9.

PROOF OF THEOREM 10. Let $A = Z_{e_1} \times \cdots \times Z_{e_b}$ and let

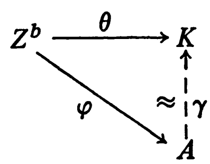
$$\varphi: Z \times \cdots \times Z \to Z_{e_1} \times \cdots \times Z_{e_b}$$

be the natural projection. Define a subgroup C of Aut(A) by

$$C = \{\tilde{g} \in \mathrm{Aut}(A) | \text{there is some } g \in \mathrm{GL}(b, Z) \text{ such that } \tilde{g}\varphi = \varphi g\}.$$

Note that C is canonically defined, that C depends not on θ but only on the isomorphism class of $\theta(Z^b)$ (namely A), and that $C = \mathrm{Aut}(A)$ if A has only elements of order 1, 2, 3, 4, or 6.

Now θ and φ determine γ uniquely:



Then γ induces an isomorphism $\bar{\gamma}$: Aut$(A) \to$ Aut(K), where we define $K = \theta(Z^b)$. Let

$$i: N(K)/C(K) \to \mathrm{Aut}(K)$$

be the usual injection. Define

$$\psi: N(K)/C(K) \to \mathrm{SL}(b, Z_2)$$

using i and the homomorphism $K \to^{\gamma^{-1}} A \to^{\delta} Z_2^b$ where δ is the natural map

$$\delta: A \to A \otimes Z_2 \approx Z_2^b.$$

Clearly ψ is 1-1 if $A = Z_2^b$. Define $H(K) = i^{-1} \circ \bar{\gamma}(C)$.

Now we will show that $G(M) = \psi(H(K))$. Consider the following diagram:

$$\begin{array}{ccccccc} Z^b & \xrightarrow{\varphi} & A & \xrightarrow{\gamma} & K & \xrightarrow{\delta\cdot\gamma^{-1}} & Z_2^b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle I_p} & & \downarrow{\scriptstyle \bar{g}} \\ Z^b & \xrightarrow{\varphi} & A & \xrightarrow{\gamma} & K & \xrightarrow{\delta\cdot\gamma^{-1}} & Z_2^b \end{array}$$

Given I_p for $p \in N(K)$, we get $\bar{g}$ such that the right-hand square commutes and we get $\tilde{g}$ such that the middle square commutes. Then g exists such that the left-hand square commutes if and only if $\tilde{g} \in C$. By definition, $\bar{g} \in G(M)$ if and only if g exists. Thus $G(M) = \psi(H(K))$. This completes the proof of Theorem 10.

7. *Proof of Theorem* 11. To establish Theorem 11, it will be useful to have an alternative definition for "$\mu' \sim \mu$ via (g, h)", which we now develop. This definition will also be used in constructing examples.

Let $\mu \in H^2_\theta(Z^b, Z^a)$ and $\mu' \in H^2_{\theta'}(Z^b, Z^a)$. We have

$$1 \to Z^a \xrightarrow{\alpha} \Gamma(\theta, \mu) \xrightarrow{\beta} Z^b \to 1.$$

Choose $t_1, \ldots, t_b$ generating Z^b, $u_1, \ldots, u_b \in \Gamma(\theta, \mu)$ such that $\beta(u_i) = t_i$ for all i, and $y_{rt} \in Z^a$ such that $y_{rt} u_r u_t = u_t u_r$ for all r, t. We also have

$$1 \to Z^a \xrightarrow{\alpha'} \Gamma(\theta', \mu') \xrightarrow{\beta'} Z^b \to 1.$$

Choose $u'_1, \ldots, u'_b \in \Gamma(\theta', \mu')$ such that $\beta'(u'_i) = g(t_i)$ for all i and $y'_{rt} \in Z^a$ such that $y'_r t u'_r u'_t = u'_t u'_r$.

Recall that we may identify μ with the $b(b-1)/2$-tuple $[(y_{rt})] \in H_\theta$, and likewise μ' with $[(y'_{rt})] \in H_{\theta'}$. (See Mac Lane [**18**, p. 110] or Morgan [**20**, p. 23].)

LEMMA 15. *$\mu' \sim \mu$ via (g, h) if and only if* $[(h(y_{rt}))] = [(y'_{rt})]$ *in* $H_{\theta'}$.

The proof of this result is a direct computational argument. For details, see Morgan [**20**, pp. 58–62].

PROOF OF THEOREM 11. We have $1 \to Z^a \to^\alpha \Gamma \to^\beta Z^b \to 1$ with $Z^b = \{t_1, \ldots, t_b\}$ and $\Gamma = \{v_1, \ldots, v_a; u_1, \ldots, u_b\}$ where $\alpha(Z^a) = \{v_1, \ldots, v_a\}$ and $\beta(u_i) = t_i$. Also $y_{rt} u_r u_t = u_t u_r$ where $y_{rt} \in \alpha(Z^a)$. Then $\Gamma = \Gamma(\theta, \mu)$, where $\theta \colon Z^b \to \mathrm{GL}(a, Z)$ is given by letting $t_1, \ldots, t_b$ act on $\alpha(Z^a)$ by $t_i(v_j) = u_i v_j u_i^{-1}$ and $\mu = [(y_{rt})] \in H_\theta$.

Theorem 11 now follows using Theorem 9 and the following lemma. Let $B_i = \theta(t_i)$, $s = \mathrm{order}(\mu)$, and E_{jk} be the $b \times b$ matrix everywhere zero except for a 1 in the (j, k)th place.

LEMMA 16. *Suppose the order of B_j is odd for some j. Then $g = I + m \cdot E_{jk}$ lifts to* $\mathrm{Aut}(\Gamma)$ *for all k, where* $m = \mathrm{order}(B_j) \cdot s = m_j \cdot s$.

$$g = \left[\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 1 & & m \\ & & & \cdot & \\ & & & & \cdot \end{array}\right] \begin{array}{l} \\ \\ j \\ \\ \\ \end{array}$$
$$\qquad\qquad\qquad\qquad k$$

Thus $g(t_i) = t_i$ *if* $i \neq k$, *and* $g(t_k) = t_k \cdot t_j^m$.

COROLLARY. *If the holonomy group has odd order, then all* $g \in \mathrm{SL}(b, Z_2)$ *lift to* $\mathrm{Aut}(\Gamma)$.

PROOF. We have seen in the proof of Theorem 1, part 2, that the order of μ divides the order of the holonomy group.

PROOF OF LEMMA 16. This is a straightforward computational argument using Theorem 3 and Lemma 15. See Morgan [**20**, pp. 63–65] for details.

V. Examples.

1. *Introduction.* This section contains a complete list of all compact 3-, 4-, and 5-dimensional flat solvmanifolds M (except those 5-dimensional M with first betti number 1). Also included is a count of the noncompact W over these M, if $\pi^*: H^2(T^b, Z_2) \to H^2(M, Z_2)$ is 1-1, where $\pi: M \to T^b$ is the canonical torus fibration for M. (For an actual list of the W over these M, see Morgan [**20**, §VII].)

2. *Explanation of the listing to follow.* In the list below, all compact manifolds are presented as follows. We know $M(\theta, \mu) = R^n/\Gamma(\theta, \mu)$, where $\Gamma(\theta, \mu)$ is a uniform torsion free Bieberbach group and $\Gamma(\theta, \mu) \approx \pi_1(M(\theta, \mu))$. It will suffice to describe the $\Gamma(\theta, \mu)$. These groups are always given by a short exact sequence

$$1 \to Z^a \xrightarrow{\alpha} \Gamma(\theta, \mu) \xrightarrow{\beta} Z^b \to 1$$

where b is the first betti number of $M(\theta, \mu)$. Then

$$\Gamma(\theta, \mu) = \{v_1, \ldots, v_a; u_1, \ldots, u_b\},$$

where

$$\alpha(Z^a) = \{v_1, \ldots, v_a\}$$

and

$$\beta(v_1^{n_1} \cdots v_a^{n_a} u_1^{m_1} \cdots u_b^{m_b}) = t_1^{m_1} \cdots t_b^{m_b},$$

where Z^b is generated by $t_1, \ldots, t_b$. Since $\Gamma(\theta, \mu)$ is a torsion free Bieberbach group, it is natural to define it as a discrete subgroup of $\mathrm{GL}(n + 1, R)$, where $n = a + b$, consisting of matrices of the form

$$\left(\begin{array}{c|c} A & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ \hline 0 & 1 \end{array}\right)$$

Given $\theta\colon Z^b \to \mathrm{GL}(a, Z)$ such that $\theta(Z^b)$ is a finite abelian group of rank r, we choose a basis $t_1, \ldots, t_b$ for Z^b such that $\theta(t_{b-r+1}), \ldots, \theta(t_b)$ generate $\theta(Z^b)$ and $\theta(t_i) = I$ for $i = 1, \ldots, b - r$. Let $B_i = \theta(t_i)$ for all i. Let the order of $\mu \in H^2_\theta(Z^b, Z^a)$ be s. Define

$$u_i = \left(\begin{array}{cc|c} B_i & 0 & M_i \\ \hline 0 & I & N_i \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right)$$

where N_i is the $a \times 1$ column matrix with 1 in the ith row and 0 elsewhere. The M_i are chosen depending on μ. If all the $M_i = 0$, then M is split. Define

$$v_j = \left(\begin{array}{c|c} I & \begin{matrix} L_j \\ \hline 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right)$$

where L_j is the $b \times 1$ column matrix with s in the jth row and 0 elsewhere. We may identify these v_j and u_i with the generators for $\Gamma(\theta, \mu)$ as described above. For more details on this construction, see Morgan [**20**, proof of Theorem 1]. In the presentation of the following 3-, 4-, and 5-dimensional compact manifolds, we will give $B_1, \ldots, B_b$, $M_1, \ldots, M_b$, and s for each $M(\theta, \mu)$. These completely describe the manifold. The proof that the list is complete and correct is given in Morgan [**20**, §VIII].

3. *Comments on the noncompact manifolds.* The list below also contains a count of the noncompact W over all compact M for which π^* is 1-1, where $\pi\colon M \to T^b$ is the canonical torus fibration and $\pi^*\colon H^2(T^b, Z_2) \to H^2(M, Z_2)$. It turns out that, with only a few exceptions, π^* is 1-1 only if the holonomy group has odd order or the solvmanifold M is split. Proposition 12 gives necessary and sufficient conditions on $B_1, \ldots, B_b$, $M_1, \ldots, M_b$, and s for π^* to be 1-1.

4. *List of groups.* The following list of finite abelian subgroups of GL(2, Z)

and GL(3, Z) will be useful. It contains those subgroups $G \subseteq \mathrm{GL}(a, Z)$ given by Newman [22], Seligman [23], and Tahara [24] without fixed points for $a = 2$ and 3.

Finite abelian subgroups of GL(2, Z).

$$g_1 = \{-I\} \cong Z_2, \quad g_2 = \left\{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right\} \cong Z_3,$$

$$g_3 = \left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} \cong Z_4, \quad g_4 = \left\{\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right\} \cong Z_6,$$

$$h_1 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -I\right\} \cong Z_2 \times Z_2,$$

$$h_2 = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, -I\right\} \cong Z_2 \times Z_2.$$

Finite abelian subgroups of GL(3, Z).

$$G_1 = \{-I\} \cong Z_2, \quad G_2 = \left\{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\} \cong Z_4,$$

$$G_3 = \left\{\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\} \cong Z_4, \quad G_4 = \left\{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}\right\} \cong Z_6,$$

$$G_5 = \left\{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}\right\} \cong Z_6, \quad G_6 = \left\{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}\right\} \cong Z_6,$$

$$H_1 = \left\{\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, -I\right\} \cong Z_2 \times Z_2,$$

$$H_2 = \left\{\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}\right\} \cong Z_2 \times Z_2,$$

$$H_3 = \left\{ \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & 0 & -1 \\ & -1 & 0 \end{bmatrix} \right\} \cong Z_2 \times Z_2,$$

$$H_4 = \left\{ \begin{bmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}, -I \right\} \cong Z_2 \times Z_2,$$

$$H_5 = \left\{ \begin{bmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \right\} \cong Z_2 \times Z_2,$$

$$H_6 = \left\{ \begin{bmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \cong Z_2 \times Z_2,$$

$$H_7 = \left\{ \begin{bmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{bmatrix}, -I \right\} \cong Z_4 \times Z_2,$$

$$H_8 = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, -I \right\} \cong Z_4 \times Z_2,$$

$$H_9 = \left\{ \begin{bmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 1 \end{bmatrix}, -I \right\} \cong Z_6 \times Z_2.$$

5. *The lists of flat solvmanifolds.* The format of a typical entry in the list below is given as follows. Consider

$$\begin{array}{llll} 13. & T^2 \to & M \to & T^3 \\ & & h_1 & Z_2 \times Z_2 \\ & & & s = 2, \quad M_1 = 0, \quad M_2 = 0, \quad M_3 = [0, 1]. \end{array}$$

(This is the 13th listed 5-dimensional manifold.)

The first line gives the canonical torus fibration for M. The first betti number is 3.

The second line gives the holonomy group h_1 with its isomorphism type $Z_2 \times Z_2$. From this we get

$$B_1 = \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}, \quad B_2 = \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}, \quad B_3 = \boxed{\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}}.$$

In general, the generators for the holonomy group are taken to be $B_{b-r+1}, \ldots, B_b$ where r is the rank of the holonomy group and b is the first betti number of M. $B_1, \ldots, B_{b-r}$ are set equal to I.

The third line gives s and M_1, M_2, M_3. While M_i is actually a column matrix, we will always write the transpose. Thus

$$M_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

but we write $M_3 = [0, 1]$. We will generally omit mention of M_i if it is the zero matrix. We sometimes omit mention of s if it equals 1.

If π^* is 1-1, we write

n noncompact

in the third line also, where n is the number of noncompact W over M.

If the word "noncompact" does not appear in the third line, then π^* is not 1-1 for this M.

Finally, if one of the above lines is omitted, then it is taken to be the same as for the previously listed manifold. Using these conventions, the above description is actually listed as

13. $s = 2, \quad M_3 = [0, 1]$

since the first two lines are the same as for manifold 12 and $M_1 = M_2 = 0$.

The three-dimensional compact flat solvmanifolds.

1.	$M = T^3$		5 noncompact
2.	$T^1 \to$	$M \to$	T^2
		(-1)	Z_2
			$s = 1$, 6 noncompact
3.			$s = 2, M_1 = [1]$
4.	$T^2 \to$	$M \to$	T^1
		g_1	Z_2
			$s = 1$, 2 noncompact
5.		g_2	Z_3
6.		g_3	Z_4
7.		g_4	Z_6

SUMMARY: 7 *compact manifolds*:
1 with first betti number 3;
2 with first betti number 2;
4 with first betti number 1.

Wolf [25] lists 10 compact flat manifolds of dimension 3. It is easy to pick out the 7 solvmanifolds in this list. In Wolf's notation, they are $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ on p. 122. (See the discussion of holonomy on pp. 117 and 120 of [25].)

The four-dimensional compact flat solvmanifolds.

1.	$M = T^4$		7 noncompact
2.	$T^1 \to$	$M \to$	T^3
		(-1)	Z^2
			$s = 1$, 11 noncompact
3.			$s = 2, M_1 = [1]$
4.	$T^2 \to$	$M \to$	T^2
		g_1	Z_2
			$s = 1$, 6 noncompact
5.			$s = 2, M_1 = [-1, 0]$
6.		g_2	Z_3
			$s = 1$, 4 noncompact
7.			$s = 3, M_1 = [-1, -1]$, 4 noncompact
8.		g_3	Z_4
			$s = 1$, 6 noncompact
9.			$s = 2, M_1 = [-1, -1]$
10.		g_4	Z_6
			$s = 1$, 6 noncompact
11.		h_1	$Z_2 \times Z_2$
			$s = 1$, 6 noncompact
12.			$s = 2, M_2 = [1, 0]$
13.			$s = 2, M_2 = [1, 1]$
14.		h_2	$Z_2 \times Z_2$
			$s = 1$, 6 noncompact
15.			$s = 2, M_2 = [0, -1]$
16.	$T^3 \to$	$M \to$	T^1
		G_1	Z_2
			$s = 1$, 2 noncompact
17.		G_2	Z_4
18.		G_3	Z_4
19.		G_4	Z_6
20.		G_5	Z_6
21.		G_6	Z_6

SUMMARY: 21 *compact manifolds*:

1 with first betti number 4;

2 with first betti number 3;
12 with first betti number 2;
6 with first betti number 1.

Wolf [25] states, without listing them, that there are 75 compact flat manifolds of dimension 4.

The five-dimensional compact flat solvmanifolds with first betti number greater than one.

1.	$M = T^5$		8 noncompact
2.	$T^1 \to$	$M \to$	T^4
		(-1)	Z_2
			$s = 1$, 17 noncompact
3.			$s = 2, M_1 = [1]$
4.	$T^2 \to$	$M \to$	T^3
		g_1	Z_2
			$s = 1$, 11 noncompact
5.			$s = 2, M_1 = [-1, 0]$
6.			$s = 2, M_1 = [-1, 0], M_2 = [0, -1]$
7.		g_2	Z_3
			$s = 1$, 5 noncompact
8.			$s = 3, M_1 = [-1, -1]$, 5 noncompact
9.		g_3	Z_4
			$s = 1$, 11 noncompact
10.			$s = 2, M_1 = [-1, -1]$
11.		g_4	Z_6
			$s = 1$, 11 noncompact
12.		h_1	$Z_2 \times Z_2$
			$s = 1$, 19 noncompact
13.			$s = 2, M_3 = [0, 1]$
14.			$s = 2, M_2 = [-1, 0], M_3 = [0, 1]$
15.			$s = 2, M_1 = [-1, 0]$
16.			$s = 2, M_1 = [-1, 0], M_3 = [0, 1]$
17.			$s = 2, M_1 = [-1, -1], M_2 = [0, 1], M_3 = [0, 1]$
18.		h_2	$Z_2 \times Z_2$
			$s = 1$, 19 noncompact
19.			$s = 2, M_1 = [-1, -1]$, 34 noncompact
20.			$s = 2, M_2 = [-1, 0]$
21.			$s = 2, M_1 = [-1, -1], M_2 = [-1, 0]$

22.	$T^3 \to$	$M \to$	T^2
		G_1	Z_2
			$s = 1$, 6 noncompact
23.			$s = 2, M_1 = [-1, 0, 0]$
24.		G_2	Z_4
			$s = 1$, 6 noncompact
25.			$s = 2, M_1 = [-1, 0, 0]$
26.			$s = 2, M_1 = [0, -1, -1]$
27.			$s = 2, M_1 = [-1, -1, -1]$
28.		G_3	Z_4
			$s = 1$, 6 noncompact
29.			$s = 2, M_1 = [-1, 2, 2]$, 6 noncompact
30.			$s = 4, M_1 = [-1, 2, 2]$
31.		G_4	Z_6
			$s = 1$, 6 noncompact
32.			$s = 2, M_1 = [-1, 0, 0]$
33.		G_5	Z_6
			$s = 1$, 6 noncompact
34.			$s = 2, M_1 = [-1, 0, 0]$
35.		G_6	Z_6
			$s = 1$, 6 noncompact
36.			$s = 2, M_1 = [-1, -1, 1]$
37.		H_1	$Z_2 \times Z_2$
			$s = 1$, 8 noncompact
38.			$s = 2, M_1 = [0, 0, -1]$
39.			$s = 2, M_1 = [-1, 0, 0]$
40.			$s = 2, M_1 = [-1, 0, -1]$
41.		H_2	$Z_2 \times Z_2$
			$s = 1$, 4 noncompact
42.			$s = 2, M_2 = [1, 0, 0]$
43.			$s = 2, M_2 = [1, 1, 0]$
44.			$s = 2, M_1 = [0, 0, -1], M_2 = [1, 1, 0]$
45.		H_3	$Z_2 \times Z_2$
			$s = 1$, 6 noncompact
46.			$s = 2, M_1 = [-1, 0, 0]$
47.			$s = 2, M_2 = [0, 1, 0]$
48.			$s = 2, M_1 = [-1, 0, 0], M_2 = [0, 1, 0]$
49.		H_4	$Z_2 \times Z_2$
			$s = 1$, 8 noncompact
50.			$s = 2, M_1 = [-1, 0, 0]$

51.	H_5	$Z_2 \times Z_2$ $s = 1$, 4 noncompact
52.		$s = 2, M_2 = [1, 0, 0]$
53.	H_6	$Z_2 \times Z_2$ $s = 1$, 4 noncompact
54.		$s = 2, M_1 = [0, -1, -1], M_2 = [0, -1, 1]$ 4 noncompact
55.		$s = 4, M_1 = [0, -1, -1], M_2 = [0, -1, 1]$
56.	H_7	$Z_4 \times Z_2$ $s = 1$, 8 noncompact
57.		$s = 2, M_1 = [-1, 0, 0]$
58.		$s = 2, M_1 = [0, -1, 0]$
59.		$s = 2, M_1 = [-1, -1, 0]$
60.	H_8	$Z_4 \times Z_2$ $s = 1$, 8 noncompact
61.		$s = 2, M_1 = [0, 0, -1]$
62.	H_9	$Z_6 \times Z_2$ $s = 1$, 8 noncompact
63.		$s = 2, M_1 = [-1, 0, 0]$

SUMMARY: 63 *compact manifolds*:
1 with first betti number 5;
2 with first betti number 4;
18 with first betti number 3;
42 with first betti number 2.

Remark on the first betti number 1 *case*. By Theorem 5, the similarity classes of 4×4 integral matrices A of finite order without fixed points are in 1-1 correspondence with the diffeomorphism classes of five dimensional compact flat solvmanifolds with first betti number 1. These manifolds are all split with 2 associated noncompact W.

BIBLIOGRAPHY

1. L. Auslander, *Some compact solvmanifolds and locally affine spaces*, J. Math. Mech. **7** (1958), 963–975. MR **21** #93.

2. ———, *Fundamental groups of compact solvmanifolds*, Amer. J. Math. **82** (1960), 689–697. MR **23** #A242.

3. ———, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Ann. of Math. (2) **71** (1960), 579–590. MR **22** #12161.

4. ———, *An exposition of the structure of solvmanifolds*. I: *Algebraic theory*, Bull. Amer. Math. Soc. **79** (1973), 227–285.

5. L. Auslander and M. Auslander, *Solvable Lie groups and locally Euclidean Riemannian spaces*, Proc. Amer. Math. Soc. **9** (1958), 933–941. MR **21** #2021.

6. L. Auslander and M. Kuranishi, *On the holonomy group of locally Euclidean spaces*, Ann. of Math. (2) **65** (1957), 411–415. MR **19**, 196.

7. L. Auslander and R. H. Szczarba, *Characteristic classes of compact solvmanifolds*, Ann. of Math. (2) **76** (1962), 1–8. MR **25** #4547.

8. ______, *Vector bundles over tori and noncompact solvmanifolds*, Amer. J. Math. **97** (1975), 260–281.

9. ______, *On free nilmanifolds and their associated non-compact solvmanifolds*, Duke J. Math. **42** (1975), 357–369.

10. L. Auslander and R. Tolimieri, *Splitting theorems and the structure of solvmanifolds*, Ann. of Math. (2) **92** (1970), 164–173. MR **43** #2733.

11. L. Bieberbach, *Über die Bewegungsguggen der Euklidischen Raume*. I, Math. Ann. **70** (1911), 297–336.

12. ______, *Über die Bewegungsguggen der Euklidischen Raume*. II, Math. Ann. **72** (1912), 400–412.

13. L. S. Charlap, *Compact flat Riemannian manifolds*. I, Ann. of Math. (2) **81** (1965), 15–30. MR **30** #543.

14. L. S. Charlap and A. T. Vasquez, *Compact flat Riemannian manifolds*. II: *The cohomology of Z_p-manifolds*, Amer. J. Math. **87** (1965), 551–563. MR **32** #6491.

15. ______, *Compact flat Riemannian manifolds*. III: *The group of affinities*, Amer. J. Math. **95** (1973), 471. MR **49** #1431.

16. C. Chevalley, *On the topological structure of solvable groups*, Ann. of Math. (2) **42** (1941), 668–675. MR **3**, 36.

17. C. W. Curtis and I. Riener, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962. MR **26** #2519.

18. S. Mac Lane, *Homology*, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #122.

19. A. Mal'cev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32; English transl., Amer. Math. Soc. Transl. (1) **9** (1962), 276–307. MR **10** #507.

20. A Morgan, *The classification of flat solvmanifolds*, Ph.D. Dissertation, Yale Univ., 1975.

21. G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2) **60** (1954), 1–27.

22. M. Newman, *Integral matrices*, Academic Press, New York, 1972. MR **49** #5038.

23. G. Seligman, *On the two-dimensional algebraic groups*, Scripta Math. **29** (1973), 453–465.

24. K. Tahara, *On the finite subgroups of* GL(3, Z), Nagoya Math. J. **41** (1971), 169–209.

25. J. A. Wolf, *Spaces of constant curvature*, Publish or Perish, Boston, Mass., 1973. MR **49** #7958.

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